Problem 1. Repeated games

Consider two wireless transmitters, which can transmit at high power (H) or at low power (L). The transmission rate that they can achieve depends on both their transmission power and on the other player's decision, according to the following payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} L & H \end{array} \\ \begin{array}{c} L \\ H \end{array} & \begin{bmatrix} (2,2) & (0,3) \\ (3,0) & (1,1) \end{bmatrix} \end{array}$$

They both want to maximize their transmission rate, and they have to transmit T packets. They can decide the transmission power for each packet, and they have to make their decision simultaneously, without knowing what the other transmitter will do. Therefore, the action space for each player is

$$\Gamma_i = \underbrace{\{L,H\} \times ... \times \{L,H\}}_{T \text{ times}}$$

and each action is a sequence of T symbols such as $\gamma_i = \underbrace{(L, H, ..., H)}_{T \text{ symbols}}$.

The payoff for an action γ_i is the sum of the payoffs for transmission of each packet.

a) How many actions does each player have?

Solution: Each player has 2 actions for each stage of the game. Thus, the number of possible actions in T stage is 2^T.

b) Draw the matrix representation of the game for $T = 2$ packets.

Solution: The resulting static game is

	(LL)	(LH)	(HL)	(HH)
(LL)	$(4,4)$	$(2,5)$	$(2,5)$	$(0,6)$
(LH)	$(5,2)$	$(3,3)$	$(3,3)$	$(1,4)$
(HL)	$(5,2)$	$(3,3)$	$(3,3)$	$(1,4)$
(HH)	$(6,0)$	$(4,1)$	$(4,1)$	$(2,2)$

c) Find a Nash Equilibrium for the game with $T = 2$ packets

Solution: We can verify that the only Nash equilibrium is (HH, HH), since for both players, HH is a dominant strategy.

d) What is the Nash Equilibrium for $T = 200$?

Solution: For $T = 200$, we check if playing always H is a Nash Equilibrium. It is. In fact, if at any time t player i plays L, then it could increase his payoff by switching to H.

e) Now assume each player can revise its action after seeing the outcome of the game at the previous stage. Consider the trigger strategy: a player i playing the trigger strategy starts by playing L, and keeps playing L. However, if the other player plays H, then player i changes to H in the next stage, and keeps playing H for the rest of the game. What would be the outcome of the game under the trigger strategy? Is this a Nash equilibrium strategy? Justify your answer.

Solution: If both players play the trigger strategy, they end up playing always L. This yields the outcome $(2T, 2T)$, as players get a payoff of $(2,2)$ at every stage. It is not a Nash equilibrium. In fact, a player could deviate from the trigger strategy and play H at the last stage (T), therefore obtaining a larger outcome (by 1), while the other players sticks to the same strategy (and to the exact same sequence $L ... L$).

In contrast to the case above, now they can observe the other player's strategy after they both transmit a packet (after each stage of the game).

f) How many strategies does each player have?

Solution: Player 1 has two pure strategies for each of their info sets, and Player 2 has the same number of pure strategies as player 1. Note that by induction, we can show that at stage t of the game, player 1 has 4^{t-1} info sets. Hence, both players have $2^{\sum_{t=0}^{T} 4^{t-1}}$ pure strategies. It may also be helpful to recall Problem 1a): Here $k = 2$ and $r = 4^{t-1}$ at stage t.

g) Draw the game in extensive form for $T = 2$ packets.

Solution: The game tree is shown in Figure 1.

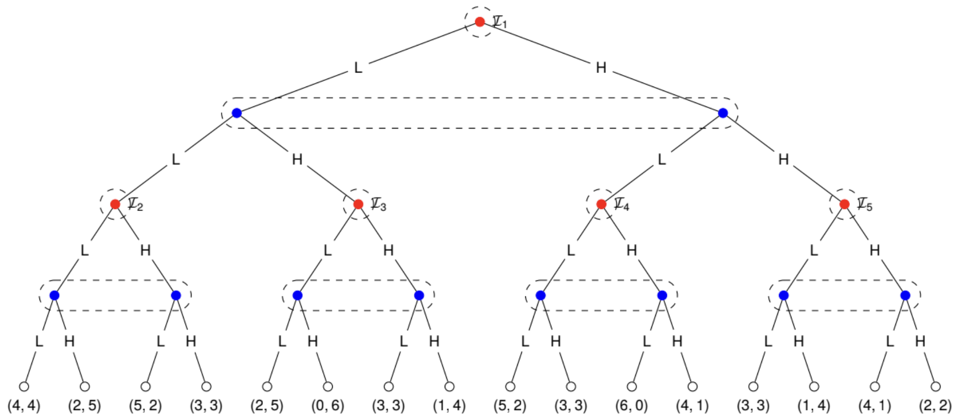

Figure 1: The game tree.

h) Determine the subgame perfect equilibrium for $T = 2$.

Solution: Observe that the payoff of each leaf of the information set $\mathcal{I}_i$ for $i = 2, 3, 4, 5$ is a constant $(c_1, c_2) \in \mathbb{R}^2$ added to the payoff corresponding to the original game's matrix. For instance, the payoff matrix for the information set $\mathcal{I}_2$, is $(2, 2)$, or the payoff corresponding to choosing (L, L) in the first stage, added to each element the original game's payoff matrix. We thus conclude that the Nash Equilibrium strategy for each second stage game is (H, H). This gives the following tree shown in Figure 2. Arguing in the same manner, we see that the payoff above is a constant $(1, 1)$ added to the original payoffs. Hence, the Nash equilibrium strategy at this stage is also (H, H). Hence, we conclude that player 1 should map each of its information sets to H in the subgame perfect equilbrium. This results in the strategy $\gamma(\mathcal{I}_i) = H$ for $i \in \{1, 2, 3, 4, 5\}$. Similarly, player 2 should map each of their information sets to H, resulting in the strategy $\sigma(\mathcal{I}_i) = H$ for $i \in \{1, 2, 3, 4, 5\}$.

i) How many information sets does each player have for $T = 200$? Determine the subgame perfect equilibrium in this case.

Solution: From part f), we find that each player has $\sum_{t=1}^{200} 4^{t-1} = \frac{4^{200}-1}{3}$ information sets, and inductively, by the previous argument in h), the subgame perfect equilibrium for $T = 200$ is to map each information set to H, for both players.

Consider now the case in which players have to transmit packets indefinitely $T = \infty$. To make the problem

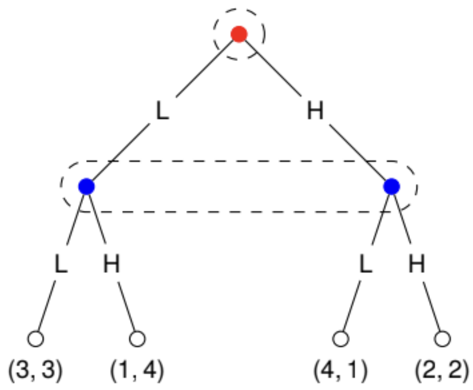

Figure 2: Tree.

well-posed we need the players' payoff to be finite. So, consider the following payoff for each player i:

$$J_i = (1-\delta)\sum_{t=0}^{T}\delta^t J_i(t)$$

where $1/2 < \delta < 1$ and $J_i(t)$ is the payoff of player i at stage (packet) $t = 1, 2, \ldots$ (as written in the matrix above).

j) Consider the strategy in which both players always play H. What is the outcome, when $T = \infty$? Is this a Nash Equilibrium strategy?

Solution: For $T = \infty$, the outcome of the strategy $H, H, \ldots$ for both players is simply

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t 1 = 1.$$

It is a Nash equilibrium strategy. In fact, deviating from H to L at any point will give you a lower outcome.

k) Consider the trigger strategy: a player i playing the trigger strategy starts by playing L, and keeps playing L. However, if the other player plays H, then player i changes to H, and keeps playing H for the rest of the game. What is the outcome, when $T = \infty$, if all players play the trigger strategy? Is this a Nash Equilibrium strategy?

Solution: If both players play the trigger strategy they will play L indefinitely, which gives an outcome of $(2, 2)$ (by repeating the exact same sum as before, but assuming a payoff of $(2, 2)$ at every stage). This time, however, the trigger strategy is a Nash equilibrium. Suppose in fact that player i plays H at time $k-1$, instead of L. Its payoff becomes

$$J_i = (1-\delta)\left(\sum_{t=0}^{k-2}\delta^t 2 + \delta^{k-1}3 + \sum_{t=k}^{\infty}\delta^t 1\right),$$

where we used that the other player will play H from step k on. Therefore the payoff for player i is going to be not better than

$$
\begin{aligned}
J_i &= (1-\delta)\left(\sum_{t=0}^{k-2} 2\delta^t + 3\delta^{k-1} + \sum_{t=k}^{\infty} \delta^t\right) \\
&= (1-\delta)\left(2\sum_{t=0}^{k-1} \delta^t + \delta^{k-1} + \sum_{t=k}^{\infty} \delta^t + \sum_{t=k}^{\infty} \delta^t - \sum_{t=k}^{\infty} \delta^t\right) \\
&= (1-\delta)\left(2\sum_{t=0}^{\infty} \delta^t + \delta^{k-1} - \sum_{t=k}^{\infty} \delta^t\right) \\
&= 2 + (1-\delta)\left(\delta^{k-1} - \sum_{t=k}^{\infty} \delta^t\right) \\
&= 2 + (1-\delta)\left(\delta^{k-1} - \delta^k \sum_{t=0}^{\infty} \delta^t\right) \\
&= 2 + (1-\delta)\delta^{k-1} - \delta^k \\
&= 2 + \underbrace{\delta^{k-1}}_{<1}\underbrace{(1-2\delta)}_{<1} \\
&< 2
\end{aligned}
$$

which is worse than the outcome of the trigger strategy (i.e., 2) because $\delta > 1/2$.

For the next exercise, you would need to use the Hespanha, Noncooperative Game Theory, 2015, available online from EPFL library website (link provided in the course Moodle cite).

Problem 2. Behavioral strategies in zero-sum games

Solve Exercise 8.3 from Hespanha book.